International Journal of Theoretical Physics, Vol. 39, No. 4, 2000

Discrete Space-Time

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Received January 2, 1999

A variant of the causal set hypothesis is discussed. Discrete space-time is an oriented graph, and the vertexes of the graph are world points. The oriented edges are elementary causal connections. The graph is the deepest level of matter. The vertexes and the edges are elementary objects and have no internal structure. All information consists in the structure of the graph, which is described by sums of paths. Grassmann variables and integrals of Grassmann variables are used for summing paths. The sums of paths define complex amplitudes, which correspond to each pair of vertexes. The complex amplitudes of all pairs of vertexes comprise a Hermitian amplitude matrix.

1. INTRODUCTION

A variant of the causal set hypothesis is discussed in this paper. According this hypothesis, space-time is a discrete, locally finite, partially ordered set (Bombelli *et al.*, 1987; Sorkin, 1991). The aim of this paper is to find a mathematical description of the causal set that is connected with quantum theory in the sense of the correspondence principle. Graph theory is used. Some preliminary results are given in Krugly (1998).

2. MODEL

Suppose space-time is an oriented graph (the edges of the graph have an orientation). Graph theory is presented in Ore (1962). The set of events is discrete. Some pairs of events are connected by discrete causal connections. A fragment of such a graph is represented in Fig. 1. The vertexes are world points. The represented fragment contains nine vertexes. In this paper the vertexes always have numbers. The oriented edges are elementary causal

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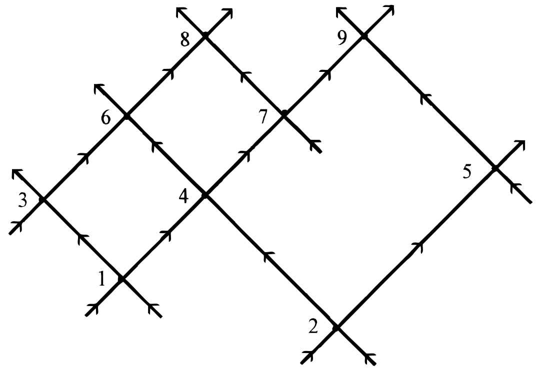

Fig. 1. An example of a graph.

connections. An edge is directed from a vertex-cause to a vertex-effect. In Fig. 1 some edges are linked with only one vertex because the paired vertex lies outside the figure. We do not discuss the problem of the finiteness or infinity of the graph of the universe. In any case we can consider only finite fragments and take into account the rest of the graph in an approximate way.

A set of edges is called a sequence if every two neighboring edges have a common vertex. A sequence is called oriented if all edges of the sequence are included in the direction of the orientation. Two vertexes a and b have a causal connection if there is an oriented sequence between them. The numbers of the vertexes are denoted by lowercase Latin letters. Vertex a is the cause of vertex b if vertex a is the initial vertex of this sequence. By S_{ab} we denote the sequence between vertexes a and b. An initial vertex is placed first in designations. S_{ab} is called cyclic if the vertexes a and b coincide. The causality principle is the prohibition of oriented cyclic sequences.

Suppose the causality principle is fulfilled. The all-vertex set divides into three subsets for each vertex a. The first set is the causes of the vertex a. The second set is the consequences of the vertex a. The third set does not connect with the vertex a by causal connections. In Fig. 1, vertexes 1, 2, and 4 are the causes of vertex 7, the vertexes 8 and 9 are effects of vertex 7, and the vertexes 3, 5, and 6 do not connect by causal connections with the vertex 7 in the considered fragment of the graph.

Suppose the graph is the deepest level of matter. Vertexes and edges are elementary objects and have no internal structure. All the vertexes are identical and so are all edges. All information consists in the structure of the graph. The graph makes up space-time. Particles are fragments of the graph with some symmetry.

The causality principle is the unique restriction on the structure of the graph in this paper. The graph satisfies the following conditions (however, the results in this paper do not depend on these conditions):

1. Suppose the number of incident edges directed to a vertex is equal to the number of incident edges directed from this vertex. This postulate is called the fundamental law of conservation.

2. The minimum number of interacting objects is equal to two. Suppose each vertex has two incident edges directed to this vertex. This postulate is called the binary principle. Then each vertex has two incident edges directed from this vertex. The graph in Fig. 1 is such a graph.

In this paper vertexes are discrete analogues of points of continuous space-time. It should be possible to formulate this model in terms of binary alternatives. The graph in Fig. 1 is an analogue of the chessboard in Fig. 2 in Finkelstein (1969). The main ideas of this approach are discussed in von Weizäcker (1975, 1977).

3. SUMS OF SEQUENCES

Consider the description of a graph's structure by analogy to quantum theory. The available framework is some sum-over-histories (Sorkin, 1991). This sum-over-histories is used to construct amplitudes. The path integral is the mathematical basis of quantum theory. In the case of a graph the analogy of a path is a sequence and the analogy of a path integral is a sum of sequences.

By G denote a fragment of a graph. Consider an incidence matrix $M(G)$. A matrix element m_{ab} is equal to 1 if there is an oriented edge between an initial vertex a and a final vertex b. If there is more than one oriented edge between vertexes a and b, the element m_{ab} is equal to the number of these edges. Otherwise $m_{ab} = 0$. $M(G)$ completely describes the structure of G.

By $S_{ab}(n)$ denote the sequence between vertexes a and b if $S_{ab}(n)$ consists of n edges. Consider the matrix $(M(G))^2$. The element $m_{ab}(2)$ of the matrix $(M(G))^2$ is equal to the number of all oriented sequences $S_{ab}(2)$. The element $m_{ab}(n)$ of the matrix $(M(G))^n$ is equal to the number of all oriented sequences $S_{ab}(n)$.

Similarly, a transposed incidence matrix $M^T(G)$ describes the fragment G', which has the same structure as G, but the direction of the edges of G' is opposite to the direction of the edges of G. The element $m_{ab}(n)$ of the matrix $(M^T(G))^n$ is equal to the number of all sequences $S_{ab}(n)$ with orientation reversed.

In the general case S_{ab} includes some edges in the direction of the orientation and some edges in the reverse direction. Consider an element $m_{ab}(n)$ of a product of n matrixes $M(G)$ and $M^T(G)$. $m_{ab}(n)$ is equal to the number of all sequences $S_{ab}(n)$ if $S_{ab}(n)$ includes an edge in the direction of the orientation or in the reverse direction according to the succession of $M(G)$ and $M^T(G)$ in the product.

The same edge can be included many times in a sequence. Consequently, the number of sequences between two vertexes in G is infinite. We can sum sequences with coefficients. The sum of all sequences S_{ab} between vertexes a and b is finite if the coefficients decrease quickly enough when the number of edges in sequences increases. We can also consider paths. A sequence is a path if each edge of the sequence is included only once. A finite fragment of a graph has only a finite number of paths. They can be summed up with coefficients that have the same magnitude by analogy to quantum theory.

Replace the elements $m_{ab} = 1$ in the incidence matrix by quantities ξ_{ab}. Let

$$\xi_{ab}\xi_{ab} = \xi_{ab}\xi_{ba} = \xi_{ba}\xi_{ab} = \xi_{ba}\xi_{ba} = 0 \tag{1}$$

A product of quantities ξ_{ab} with different indexes is not equal to 0. Thus ξ_{ab} is a generator of a one-dimensional Grassmann algebra. The theory of Grassmann algebras is presented in Fearnley-Sander (1979).

If there are k edges between vertexes a and b, then a path can include all of them. In this case replace m_{ab} in the following way:

$$m_{ab} \to \sum_{l=1}^{k} \xi_{ab(l)} \tag{2}$$

where

$$\begin{aligned} \xi_{ab(l)}\xi_{ab(f)} &= 0 \qquad \text{if} \quad l = f \\ \xi_{ab(l)}\xi_{ab(f)} &\neq 0 \qquad \text{if} \quad l \neq f \end{aligned} \tag{3}$$

Thus each edge of G corresponds to a generator of a one-dimensional Grassmann algebra. By $\Xi(G)$ denote the obtained incidence matrix. Henceforth we drop the number of an edge and denote $\xi_{ab(l)}$ by ξ_{ab}. If a path between vertexes a and b consists of n edges, then it corresponds to a product $P_{ab}(n)$ of n generators of Grassmann algebras,

$$P_{ab}(n) = \xi_{ac}\xi_{cd}\xi_{dj} \ldots \xi_{mb} \tag{4}$$

when the indexes are equal to the numbers of vertexes along this path.

The element $\xi_{ab}(n)$ of the product of n matrixes $\Xi(G)$ and $\Xi^T(G)$ is equal to the sum of all paths $P_{ab}(n)$ if these paths include the edges in the direction of the orientation or in the inverse direction according to the succession of $\Xi(G)$ and $\Xi^T(G)$ in the product.

Let us discuss the following possibility:

$$\{\xi_{ab}\xi_{cd}\}_{+} = \xi_{ab}\xi_{cd} + \xi_{cd}\xi_{ab} = 0 \tag{5}$$

In this case the elements ξ_{ab} are generators of one Grassmann algebra. The dimension of this algebra is equal to the number of edges of G. $P_{ab}(n)$ is an element of this Grassmann algebra. Condition (1) is a consequence of condition (5).

In the general case a permutation of the edges in a path does not generate a path. Consider a special case. Let a path include one circuit, i.e., a cyclic path without self-crossing. The permutation of all edges of the circuit generates a new path. In this path the edges of the circuit are included in the reverse order and direction (Fig. 2). Another part of the initial path is not changed. These paths have a different sign according to (5) and they are excluded from $\xi_{ab}(n)$ if the number k of edges of the circuit is even and $k/2$ is odd, or if k is odd and $(k - 1)/2$ is odd. Then condition (5) excludes some paths from $\xi_{ab}(n)$.

Let us consider a stronger restriction. Consider arcs. The arc is a paths without self-crossings.

Let each vertex a correspond to the quantities ξ_a and ζ_a and the edge between vertexes a and b correspond to the product $\xi_a\zeta_b$. Let an arc between vertexes a and b consist of n edges. This arc corresponds to a product $A_{ab}(n)$ of $2n$ quantities if conditions (6)–(8) hold:

$$A_{ab}(n) = \xi_a\zeta_c\xi_c\zeta_d \ldots \xi_i\zeta_b \tag{6}$$

where the indexes are equal to the numbers of the vertexes along this arc.

A vertex in an arc cannot be the beginning of edges twice. Thus, ξ_a is a generator of a one-dimensional Grassmann algebra:

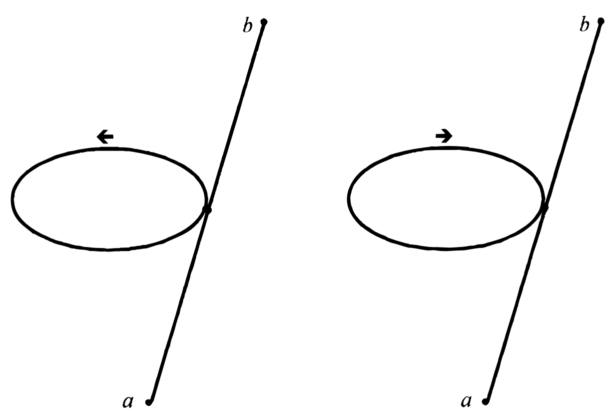

Fig. 2. An example of a path that includes one circuit. The arrow shows the direction of the circuit in the path.

$$\xi_a \xi_a = 0 \tag{7}$$

A vertex in an arc cannot be the end of edges twice. Thus ζ_a is also a generator of a one-dimensional Grassman algebra:

$$\zeta_a \zeta_a = 0 \tag{8}$$

The case of $A_{ab}(2)$ is an exception and will be considered below.

Substitute the products $n_{ab}\xi_a\zeta_b$ for the elements m_{ab} of $M(G)$. n_{ab} is equal to the number of edges between vertexes a and b. By $Z(G)$ denote the obtained matrix. Substitute the products $n_{ab}\zeta_b\xi_a$ for the elements $n_{ab}\xi_a\zeta_b$ of the matrix $Z(G)$ and transpose this matrix. By $Z'(G)$ denote the obtained matrix. $Z'(G)$ describes the fragment G'. An element $\zeta_{ab}(n)$ of a product of n matrices $Z(G)$ and $Z'(G)$ is equal to the sum of all arcs $A_{ab}(n)$ if these arcs include the edges in the direction of the orientation or in the inverse direction according to the succession of $Z(G)$ and $Z'(G)$ in the product, except the case of $A_{ab}(2)$.

The correspondence of the quantities ξ_a and ζ_a to the vertexes does not allow us to distinguish single and multiple edges. Consequently in the case of $\zeta_{ab}(2)$ the obtained sum includes a cyclic sequence that consists of one edge. This edge is included in this sequence twice: first in one direction and then in the inverse direction (Fig. 3). Consider the diagonal matrix $N(G)$ with the elements

$$\nu_{aa} = \sum_b n_{ab}\xi_a\zeta_b\xi_b\zeta_a \tag{9}$$

Consider the matrix $Z(G, 2)$

$$Z(G, 2) = Z(G)Z'(G) + Z'(G)Z(G) - N(G) \tag{10}$$

The elements of this matrix are equal to the sums of all arcs that consist of two edges.

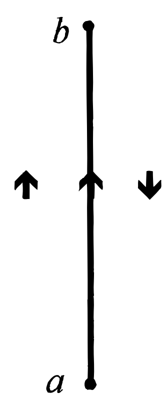

Fig. 3. A cyclic sequence that consists of one edge.

As in the case of paths, let

$$\{\xi_a \xi_b\}_+ = \xi_a \xi_b + \xi_b \xi_a = 0 \tag{11}$$

Then the quantities ξ_a are generators of one Grassmann algebra. The dimension of this algebra is equal to the number of the vertexes of G.

Similarly, let

$$\{\zeta_a \zeta_b\} = \zeta_a \zeta_b + \zeta_b \zeta_a = 0 \tag{12}$$

Then the quantities ζ_a are generators of a second Grassmann algebra. The dimension of this algebra is also equal to the number of vertexes of G.

In this case some arcs are excluded from sums as in the case of paths.

In all considered cases the sums of all paths or arcs between two vertexes are finite. The first case of paths is most interesting because the restriction is minimal. However, the results in the next section hold for all considered cases.

4. THE AMPLITUDE MATRIX

In the previous section we summed edges without coefficients. If a sum over paths corresponds to a path integral in a continuous limit, then an edge corresponds to a segment of a continuous path. In accordance with quantum theory we should multiply the element ξ_{ab} of $\Xi(G)$ by the coefficient $\exp(i\alpha)$. α is called a phase of an edge. The transposition of $\Xi(G)$ corresponds to time inversion and in this case the phase of an edge changes sign. An edge is an elementary object and has no internal structure. Consequently α is a constant. We take

$$\exp(i\alpha) = i, \qquad \alpha = \pi/2 \tag{13}$$

Let us discuss the reason for this value. Put two vertexes connected by an edge in Minkowski space-time. Put the origin of a Cartesian system at the vertex-cause. If we use a Cartesian system we can use complex coordinates (Einstein, 1910) instead of covariant and contrvariant coordinates. The coordinates of a point are (x, y, z, it). If we invert the timelike axis, the coordinates of this point are $(x, y, z, -it)$. Let us choose the Cartesian system such that the coordinates of the vertex-effect are $(0, 0, 0, it)$. The timelike axis coincides with the edge. The edge is the unit of proper time—chronous. The natural coordinates of the vertex-effect are $(0, 0, 0, i)$. I suppose the proper time interval of the edge is the amplitude of the edge. The space-time measure and the amplitude function are connected with different algebras at the sets of edges.

However, I do not use a particular value of the constant α in this paper.

Consider the sums over paths with phases. We have for paths including one edge

$$\begin{aligned}\Omega(G, 1) &= \exp(i\alpha)\ \Xi(G)\\ \Omega^{+}(G, 1) &= \exp(-i\alpha)\ \Xi^{T}(G)\end{aligned} \tag{14}$$

Consider the matrices of sums of paths including n edges with phases,

$$\begin{aligned}\Omega(G, n, j, f) &= \sum_{P} \Omega(G, 1)^{j}(\Omega^{+}(G, 1))^{f}\\ j + f &= n\end{aligned} \tag{15}$$

where we take summation over all permutations P of matrixes $\Omega(G, 1)$ and $\Omega^{+}(G, 1)$.

We obtain the expression (16) for the sums of all paths including n edges with phases,

$$\Omega(G, n) = \sum_{j=0}^{n} \Omega(G, n, j, f) \tag{16}$$

We obtain the following expression for the sums of all paths with phases:

$$\Omega(G) = \sum_{n=1}^{\infty} \Omega(G, n) \tag{17}$$

An element ω_{ab} of this matrix is equal to the sum of all paths between vertexes a and b. In expression (17) each edge of paths is a Grassmann generator multiplied by a coefficient. If the edge between vertexes c and d is included in a path in the direction of the orientation, then this edge is $\exp(i\alpha)\ \xi_{cd}$. If this edge is included in a path in the inverse direction, then this edge is $\exp(-i\alpha)\ \xi_{cd}$.

Introduce a linear operator $\hat{\xi}$, which operates on ξ_{ab} and transforms these Grassmann generators to the number 1. $\hat{\xi}$ is an integral of Grassmann variables,

$$\begin{aligned}&\hat{\xi}(a\xi_{ab} \ldots \xi_{cd} + b\xi_{ef} \ldots \xi_{kl})\\ &\quad = \int \ldots \int (a\xi_{ab} \ldots \xi_{cd})\, d\xi_{cd} \ldots \xi_{ab}\\ &\quad\quad + \int \ldots \int (b\xi_{ef} \ldots \xi_{kl})\, d\xi_{kl} \ldots d\xi_{ef} = a + b\end{aligned} \tag{18}$$

where a and b are complex numbers. Operate on $\Omega(G)$ by $\hat{\xi}$

$$\Phi(G) = \hat{\xi}\Omega(G) \tag{19}$$

The element ϕ_{ab} of the matrix $\Phi(G)$ is equal to the number of all paths with phases between vertexes a and b. $\Phi(G)$ is called the amplitude matrix and ϕ_{ab} is called the amplitude between vertexes a and b. It is possible the quantity $\phi_{ab}\phi_{ab^*}$ is some probability. This will be discussed in a further paper. Some preliminary results are given in Krugly (1998).

Thus, a complex amplitude corresponds to each pair of points of discrete space-time. It is equal to the number of all paths with phases between these points. An integral of Grassmann variables is used for summing paths. A set of amplitudes of all vertex pairs of G is an amplitude matrix of G.

An elementary particle is specified by certain quantum numbers. In the proposed discrete model a particle is some fragment of the graph and the quantum numbers should be taken from the structure of the graph. $\Phi(G)$ describes the structure of the graph. Suppose this description allows us to associate different fragments of the graph with elementary particles. Consider the properties of $\Phi(G)$.

An element of a principal diagonal of $\Phi(G)$ is a real number. A diagonal element ϕ_{aa} is a sum of paths that go out and come in at the same vertex a. They are cyclic paths. Each path is included into this sum twice, once in one direction and the second time in the inverse direction. Hence each imaginary summand has a pair with an inverse sign and they are excluded.

An off-diagonal element is complex conjugate to a symmetrical element. An off-diagonal element ϕ_{ab} is a sum of paths between vertexes a and b. A symmetrical element ϕ_{ba} is a sum of paths between vertexes b and a. These paths are the same, but with inverse direction. Hence the real parts of ϕ_{ab} and ϕ_{ba} coincide and the imaginary parts have an equal magnitude and a different sign.

Consequently an amplitude matrix is a Hermitian matrix,

$$\mathrm{Im}(\phi_{aa}) = 0, \qquad \phi_{ab} = \phi^*_{ba} \tag{20}$$

An amplitude matrix of order n corresponds to n vertexes of the graph. A fragment of the graph with some symmetry corresponds to an amplitude matrix with some special properties. If these special Hermitian matrixes of different orders correspond to particles, then the particles are described by this discrete model. This will be discussed in a further paper.

REFERENCES

Bombelli, L., Lee, J., Meyer, D., and Sorkin, R. D. (1987). *Physical Review Letters*, **59**, 521–524.

Einstein, A. (1910). Principe de relativité et ses conséquences dans la physique moderne. *Archives des Sciences Physiques et Naturelles*, *Genève, Serie* 4, **29**, 5–28, 125–144.

Fearnley-Sander, D. (1979). *American Mathematical Monthly*, **86**, 809.

Finkelstein, D. (1969). *Physical Review*, **184**, 1261–1273.

Krugly, A. L. (1998). *The Model of Discrete Space-Time*, Monolog, Moscow [in Russian].

Ore, O. (1962). *Theory of Graphs*, AMS, Providence, Rhode Island.

Sorkin, R. D. (1991). Spacetime and causal set, in *Relativity and Gravitation: Classical and Quantum. Proceedings of the SILARG VII Conference, Cocoyoc, Mexico, December, 1990*, J. C. D'Olivo *et al.*, eds., pp. 150–173.

von Weizsäcker, C. F. (1975). *Quantum Theory and the Structures of Time and Space*, Münhen, Vienna, pp. 213–230.

von Weizsäcker, C. F. (1977). *Quantum Theory and the Structures of Time and Space*, Vol. 2, Münhen, Vienna, pp. 86–112.